On the Solution and Field Pattern of Cylindrical Dielectric Resonators

Recent publications[1–6] on dielectric resonators have shown the importance and applications of these devices. The usual method of analytic solution has been to approximate the cylindrical wall with a perfect open circuit boundary (magnetic wall) and to allow decaying fields outside the flat ends.[2,4,6] These approximations have given good agreement between the calculated and measured frequencies for the mode having a fraction of a half cycle variation along the resonator length.[4] In connection with our work on the coupling of ferromagnetic and dielectric resonances,[7,8] we have occasion to investigate higher order modes in the resonator. The approach taken is different and gives interesting and more accurate results as will be discussed below.

To obtain a closer approximation, the problem of an infinitely long rod is first solved. For the TE modes, the solution inside the rod is assumed to be of the form

$$H_{iz} = AJ_n(k_{ci}r)\cos n\phi e^{j(\omega t-\beta z)} \qquad (1)$$

while that outside is assumed to be

$$H_{oz} = BH_n^{(1)}(k_{co}r)\cos n\phi e^{j(\omega t-\beta z)} \qquad (2)$$

where

$$k_{ci}^2 = \left(\frac{2\pi}{\lambda_0}\right)^2\epsilon_i - \beta^2 = k_i^2 - \beta^2, \qquad (3)$$

$$k_{co}^2 = \left(\frac{2\pi}{\lambda_0}\right)^2 - \beta^2 = k_o^2 - \beta^2, \qquad (4)$$

J_n is the Bessel Function, and

H_n is the Hankel Function.

For H_n to behave like a decaying exponential, k_{co} must be imaginary, which demands

$$\beta^2 > k_o^2. \qquad (5)$$

An immediate conclusion from (5) is that, in a resonator, there can be no mode having a uniform longitudinal field ($\beta = 0$). This can be understood physically by noting that uniform longitudinal field gives rise to a magnetic dipole which radiates strongly in free space, i.e. nonresonant.

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[1] A. Okaya and L. F. Barash, "The dielectric microwave resonator," *Proc. IRE*, vol. 50, pp. 2081–2092, October 1962.
[2] S. B. Cohn and C. W. Chandler, "Investigation of microwave dielectric-resonator filters," Rantec Corp., Calabasas, Calif., First Quarterly Rept., Contract DA36-039-AMC-02267(E), Task 5544-PM-63-91, September 1963.
[3] R. V. D'Aiello and H. J. Prager, "Dielectric resonators for microwave applications," *IEEE Trans. on Microwave Theory and Techniques (Correspondence)*, vol. MTT-12, pp. 549–550, September 1964.
[4] H. Y. Yee, "Natural resonant frequencies of microwave dielectric resonators," *IEEE Trans. on Microwave Theory and Techniques (Correspondence)*, vol. MTT-13, p. 256, March 1965.
[5] D. L. Rebsch, D. C. Webb, R. A. Moore, and J. D. Cowlishaw, "A mode chart for accurate design of cylindrical dielectric resonators," *IEEE Trans. on Microwave Theory and Techniques (Correspondence)*, vol. MTT-13, p. 468, July 1965.
[6] J. C. Sethares and S. J. Naumann, "Design of microwave dielectric resonators," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-14, pp. 2–7, January 1966.
[7] K. K. Chow and M. E. Hines, "Mode segregation effects in single crystal YIG," to be published.
[8] M. E. Hines and K. K. Chow, "Phase shifter in C-, X-, and Ku-bands using segregated-mode resonance in single crystal YIG," to be published.

Solution of Maxwell's equations using (1) and (2) gives a complete set of field expressions. However, matching is possible only for $n = 0$ if simple TE and TM solutions are assumed. Therefore, with the exception of TE_{0m} and TM_{0m} modes, all higher order modes are hybrid. For TE_{0m} modes in a rod of radius a, the secular equation is

$$k_{ci}a\frac{J_0(k_{ci}a)}{J_1(k_{ci}a)} = k_{co}a\frac{H_0^{(1)}(k_{co}a)}{H_1^{(1)}(k_{co}a)}. \qquad (6)$$

An interesting point arising from close examination of (6) is that the H_z has to reverse itself inside the rod, even for the lowest order TE_{01} mode. This is because $k_{co}aH_0^{(1)}/H_1^{(1)}$ is always negative ($k_{co}a$ being imaginary), hence J_0 and J_1 must be of different sign. This fact of field reversal inside the resonator is important if one wishes to couple to a ferrite disk embedded in the resonator, such as for longitudinal pumping.

Having solved (6) numerically, β may be obtained from (3). To find resonance conditions, one may now look upon the problem as that of a length of dielectric-filled magnetic wall waveguide, with a cutoff air-filled guide attached to both ends, as suggested by Cohn and Chandler;[2] the only difference is that β is now given by solution of (3). Following the method of Cohn and Chandler,[2] we shall investigate the case of $TE_{011+\delta}$ mode, in which the longitudinal field variation is between one-half and a full wavelength. As the flat ends approximate open circuits, the plane of symmetry (midway between the flat ends) must have maximum H_r and zero E_ϕ. That is, the plane of symmetry acts like a perfect short circuit, as shown in Fig. 1. At $z = d$, let the impedance looking into the dielectric-filled guide be Z_{in} and that looking into the air-filled guide (cutoff) be Z_a, then

$$Z_{in} = jZ_d\tan\beta d, \quad \text{and} \quad Z_a = \frac{j\omega\mu}{-\alpha} \qquad (7)$$

where Z_d = characteristic impedance for the dielectric guide

$$= -\frac{\omega\mu}{\beta}, \quad \text{and} \qquad (7a)$$

α = attenuation constant of the air guide

$$= \left[k_{ci}^2 - \left(\frac{2\pi}{\lambda_0}\right)^2\right]^{1/2}. \qquad (7b)$$

For resonance, Z_{in} and Z_a have to be complex conjugates. Therefore, from (7), resonance condition is

$$\tan\beta d = -\frac{\beta}{\alpha}, \qquad (8)$$

where β is given by (3), and is obtained by solving (6).

The theoretical curve of resonance frequency vs. resonator length is given in Fig. 1, for rutile resonators of diameter = 0.160 inch and $\epsilon = 113$. On the same figure, resonance frequencies computed by Yee's method[4] and measured frequencies are also given. It is seen that our method improves the accuracy: over the range of interest, resonance frequencies agree within 1–2 percent.

Of great interest to the problem of coupled resonances is the actual field pattern of these resonators. To plot the field experimentally, the resonators are mounted on a rotating mount in free-space with a wire loop to excite

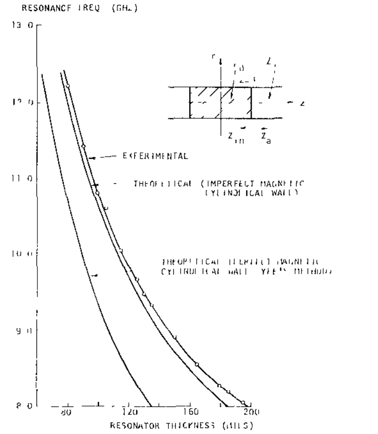

Fig. 1. Theoretical and experimental resonance frequencies for rutile resonator ($TE_{011+\delta}$ Mode).

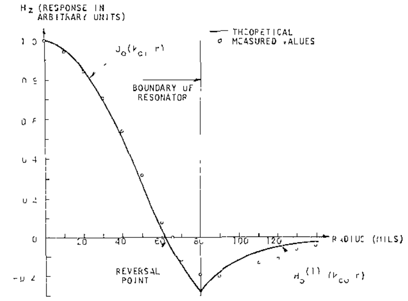

Fig. 2. Radial RF H_z distribution just outside the resonator.

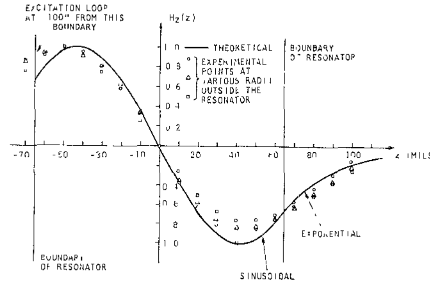

Fig. 3. Longitudinal RF H_z distribution just outside the resonator.

the TE_{0ml} mode. A loop of 0.010 inch diameter is used to probe the field just outside the resonator. The strongest and lowest frequency resonances are found to be the $TE_{011+\delta}$, the frequencies of these have been given in Fig. 1. Typical field plot is as given in Figs. 2 and 3, from which it is seen that agreement with thoretical prediction is good.

In conclusion, we have shown a new method of approximation for the solution of the dielectric resonator. The simple analysis of imperfect open-circuit boundaries brings out some interesting points readily, and greatly improves the accuracy. The actual field plot, while disagreeing with previous work,[3] nevertheless agrees well with our analysis. Finally, it should be noted that although details of analysis are shown for TE modes, identical steps may be carried out for TM and hybrid modes.

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